CALCULATION OF THE LIMITING POSITION FOR A PHASE-CHANGE BOUNDARY

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Leibenson's method has been used with fictitious boundary conditions to obtain an expression for the limiting position for the boundary to the change of state for a plane-parallel heat flux when the energy input is restricted.

In many technical problems one has to determine the position of the melting or solidification boundary for a solid, when the heat input is finite. The limiting position of this boundary is of the main interest in most cases.

Here we examine the one-dimensional Stefan's problem to show that a modified form of Leibenson's method can be used [1] to solve this problem without determining the position of the phase boundary as a function of time.

The following is the system of equations and boundary conditions corresponding to this problem:

$$\frac{\partial T_1}{\partial t} = \varkappa_1 \frac{\partial^2 T_1}{\partial x^2}, \quad 0 \leqslant x \leqslant X(t), \tag{1}$$

$$\frac{\partial T_2}{\partial t} = \varkappa_2 \frac{\partial^2 T_2}{\partial x^2}, \quad X(t) \leqslant x < \infty, \tag{2}$$

$$\left. \begin{array}{l} T_2(x,\ 0) = T_0 \quad (T_0 < T_{mp}) \\ T_2(x,\ t) \to T_0 \ \text{ for } \ x \to \infty \end{array} \right\}, \tag{3}$$

$$-\lambda_1 \left. \frac{\partial T_1}{\partial x} \right|_{x=0} = f(t), \quad \left(Q = \int_0^{t^*} f(t)\, dt < M\right), \ t^* \ll t_m, \tag{4}$$

where t_m is the time needed for the boundary X(t) to attain its limiting position X_m.

The conditions at the boundary x = X(t) are as follows:

$$T_1[X(t),\ t] = T_2[X(t),\ t] = T_{mp}, \tag{5}$$

$$-\lambda_1 \frac{\partial T_1}{\partial x} + \lambda_2 \frac{\partial T_2}{\partial x} = L\rho_2 \frac{dX(t)}{dt} \quad \text{for} \quad x = X(t). \tag{6}$$

At any time t > t* we have

$$Q = \int_0^{X(t)} c_1\rho_1 (T_1 - T_0)\, dx + L\rho_2 X(t) + \int_{X(t)}^{\infty} c_2\rho_2 (T_2 - T_0)\, dx. \tag{7}$$

Here the first term on the right is the increment in the energy in the zone $0 \leq x \leq X(t)$, while the third term is the increment in the energy in the zone $X(t) \leq x < \infty$, and the second is the energy consumed in the phase transition. Each of these terms does not exceed the total energy Q. In particular,

$$L\rho_2 X(t) < Q$$

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or

$$X(t) < \frac{Q}{L\rho_2} \leqslant \frac{M}{L\rho_2} < \infty.$$

In accordance with this we seek the maximum value for the boundary coordinate as a function of Q, T_0, and T_{mp}, i.e., $\max_t [X(t)] = X_m = X_m(T_0, T_{mp}, Q)$.

Let X_m attain the boundary at $t = t_m$; then

$$\left.\frac{dX(t)}{dt}\right|_{t=t_m} = 0. \tag{8}$$

In accordance with Leibenson's method [1], the temperature in the second zone at this instant is defined by

$$T_2(x, t_m) = T_0 + (T_{mp} - T_0)\,\mathrm{erfc}\left[\frac{x - X_m}{2\sqrt{\varkappa_2 t_m}}\right], \tag{9}$$

which corresponds to the solution to (2) in the region $X_m < x < \infty$ with the boundary immobile [see (8)] and the conditions of (3) and (5).

We have a stationary temperature distribution in the region $0 < x < X_m$ at $t = t_m$ when (8) is applied:

$$T_1(x, t_m) = A_1 + A_2 x. \tag{10}$$

With the constants A_1 and A_2 we find from conditions (5) and (6) at the boundary; then (10) takes the form

$$T_1(x) = T_{mp} - \frac{\lambda_2}{\lambda_1}\frac{(T_{mp} - T_0)}{\sqrt{\pi\varkappa_2 t_m}}(X_m - x). \tag{11}$$

In deriving (9) and (11) we have used all the conditions apart from the conditions at the boundary $x = 0$ and relationships (1) and (7); these conditions will serve to find X_m and t_m.

We substitute (9) and (11) into (7), and after integration get

$$Q = c_1\rho_1\left[(T_{mp} - T_0)X_m + \frac{\lambda_2}{2\lambda_1}\frac{(T_{mp} - T_0)}{\sqrt{\pi\varkappa_2 t_m}}X_m^2\right] - 2c_2\rho_2\sqrt{\frac{\varkappa_2 t_m}{\pi}}(T_{mp} - T_0) + L\rho_2 X_m. \tag{12}$$

We introduce the symbols

$$\begin{gathered} \Delta T = T_{mp} - T_0, \\ \nu = \frac{L\rho_2}{c_1\rho_1\Delta T} \quad \text{(Kossovich's number)}, \\ X^* = \frac{Q}{c_1\rho_1\Delta T}, \\ \alpha = \sqrt{\frac{\lambda_1 c_1\rho_1}{\lambda_2 c_2\rho_2}}. \end{gathered} \tag{13}$$

($\alpha = 1/K_\varepsilon$, where K_ε is the thermal activity number of falling ground [1].)

From (13) we put (12) in the form

$$X_m^2 + [\sqrt{\pi\varkappa_1}(\nu\alpha + \alpha)\sqrt{t_m}]X_m + 4\varkappa_1 t_m - 2X^*\sqrt{\pi\varkappa_1}\,\alpha\sqrt{t_m} = 0, \tag{14}$$

whence

$$X_m = -\sqrt{\pi\varkappa_1}(\nu\alpha + \alpha)\sqrt{t_m} + \sqrt{[\pi\varkappa_1(\nu\alpha + \alpha)^2 - 4\varkappa_1]t_m + 2X^*\sqrt{\pi\varkappa_1}\,\alpha\sqrt{t_m}}. \tag{15}$$

The + sign in front of the root means that the condition $X_m > 0$ is met.

If there is no heat flux from the second zone ($X_m \leq x < \infty$), i.e., if $\Delta T = 0$, then (12) and (14) or (15) gives the standard solution

$$X_m = \frac{Q}{L\rho_2}. \tag{16}$$

We now derive the second relationship between X_m and t_m; for this purpose we need to know the temperature at x = 0 as a function of time $\varphi(0, t)$. Then we equate the value at $t = t_m$ to the value T_1 from (11) with x = 0 to get the further relationship between X_m and t_m.

Let $\rho_1 = \rho_2$, $c_1 = c_2$, $\lambda_1 = \lambda_2$; then at $t = t_m$ the temperature distribution in the entire region $(0 \leq x < \infty)$ will be described by a smooth curve, as (6) shows. Then the temperature distribution at $t = t_m$ may be considered as equivalent to the temperature distribution at the same instant for a uniform and isotropic medium as provided by an instantaneous heat source of output $Q_1 = Q - L\rho_2 X_m$.

Then [1]

$$\varphi(x, t) = \frac{Q - L\rho_2 X_m}{c_1\rho_1\sqrt{\pi\varkappa_1 t_m}} \exp\left[-\frac{x^2}{4\varkappa_1 t}\right] + T_0. \tag{17}$$

The condition relating X_m and t_m takes the form

$$\varphi(0, t_m) = T_1(0, t_m).$$

We substitute from (1) and (17) in this expression and use (13) to get

$$\sqrt{t_m} = \frac{1}{\sqrt{\pi\varkappa_1}}\left[X^* - \left(\nu + \frac{1}{\alpha}\right)X_m\right]. \tag{18}$$

Then with $\Delta T = 0$ we have

$$X_m = Q/L\rho_2.$$

The same arguments are approximately correct for similar values of ρ_i, c_i, λ_i; a similar approach has been used for other problems in [2].

Equations (15) and (18) give us a system for X_m and t_m; we solve these to get the following expressions for X_m and t_m:

$$X_m = X^* \frac{\frac{4}{\pi}\left(\nu + \frac{1}{\alpha}\right) - (2\nu\alpha + \alpha + 1) + \sqrt{1 + \alpha^2 - \frac{4}{\pi}}}{\frac{4}{\pi}\left(\nu + \frac{1}{\alpha}\right)^2 - (2\nu^2\alpha + 2\nu\alpha + 2\nu + 1)}, \tag{19}$$

$$\sqrt{t_m} = \frac{X^*}{\sqrt{\pi\varkappa_1}} \frac{\nu\alpha + \left(\nu + \frac{1}{\alpha}\right)\left(\sqrt{1 + \alpha^2 - \frac{4}{\pi}} - 1\right)}{(2\nu^2\alpha + 2\nu\alpha + 2\nu + 1) - \frac{4}{\pi}\left(\nu + \frac{1}{\alpha}\right)^2}. \tag{20}$$

The expressions (19) and (20) for $\Delta T = 0$ become respectively

$$X_m = \frac{X^*}{\nu} = \frac{Q}{L\rho_2},$$

$$\sqrt{t_m} = \frac{X^*}{\sqrt{\pi\varkappa_1}} - \frac{\left(\nu + \frac{1}{\alpha}\right)X_m}{\pi\varkappa_1} = \frac{Q - L\rho_2 X_m}{\sqrt{\pi\varkappa_1}c_1\rho_1\Delta T} = \frac{0}{0}.$$

We eliminate the indeterminacy by means of L'Hôpital's rule to get

$$\sqrt{t_m} = \frac{X_m}{\sqrt{\pi\varkappa_1}} \frac{1 + \alpha - \frac{4}{\pi\alpha} + \sqrt{1 + \alpha^2 - \frac{4}{\pi}}}{2\alpha - \frac{4}{\pi}}. \tag{21}$$

From (10) and (20) we see that $\sqrt{1 + \alpha^2 - 4/\pi}$ indicates the lower bound to the variation in α:

$$\alpha > \sqrt{\frac{4}{\pi} - 1} = 0.522. \tag{22}$$

This restriction is a natural consequence of our assumptions about the smoothness of the temperature curve.

To conclude we consider an example. We envisage the melting of ice with an initial temperature $T_0 = -5°C$. The thermophysical parameters are $c_1 = 1$ kcal/kg-deg, $\rho_1 = 10^3$ kg/m^3, $\lambda_1 = 0.5$ kcal/m-h-deg, $T_{mp} = 0°C$, $c_2 = 0.54$ kcal/kg-deg, $\rho_2 = 900$ kg/m^3, $\lambda_2 = 1.9$ kcal/m-deg-h, $L = 80$ kcal/kg, and then $\nu = 14.4$ and $\alpha = 0.735$.

We substitute these values into (19) to get $X_m/\overline{X} = 0.98$, where $\overline{X} = Q/L\rho_2$ is the limiting coordinate for the melting front without allowance for the flux from the frozen zone ($\Delta T = 0$).

The result shows that we are correct in our assumptions about the distribution of the thermal energy at the instant when the boundary reaches its limiting position.

Then one can use (16) in rough calculations, without incurring much error.

NOTATION

c	is the specific heat of melting;
ρ	is the density;
λ	is the thermal conductivity;
$\varkappa$	is the thermal diffusivity;
L	is the latent melting heat;
T_0	is the initial temperature;
T_{mp}	is the melting point;

Subscripts

1 and 2 refer to the liquid and solid phases respectively.

LITERATURE CITED

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2. B. A. Boley, Journal of Math. and Phys., 40, No. 4, 300 (December, 1971).